Imaginary Coloring[1]

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It is a great honor for me to be published in *Geombinatorics* and to present an historical and philosophical view of important new results by Saharon Shelah and Alexander Soifer.

– Jean-Paul Delahaye

The axiom of choice, a benign matter for the non-logician, puzzles mathematicians. Today, it manifests itself in a strange way: it takes, depending on the axiom's variants, either two or infinity of colors to resolve a coloring problem.

Just as the parallels postulate seemed obvious, the axiom of choice has often been considered true and beyond discussion. The inventor of set theory, Georg Cantor (1845-1918), had used it

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several times without realizing it; Giuseppe Peano (1858-1932) used it in 1890, in working to solve a differential equation problem, consciously; but it was Ernst Zermelo (1871-1953), at the beginning of the 20th century, who identified it clearly and studied it.

This axiom states that given a set of disjoint non-empty sets, for example the set E containing the three sets {1,2}, {a, b, c}, {x, y}, then a set C exists that is composed of one element from each set in E, for example C = {2, a, y}. **Back Note 1** demonstrates two different forms of the axiom of choice and the reasons that make this axiom delicate.

The axiom of choice is independent of other axioms of set theory in the same way that the parallels postulate is independent of other axioms in geometry. Given the acceptance of these other axioms, you can still, without risk of contradiction, either accept the axiom of choice or accept its negation.

These results of so-called “relative consistency” were proven in 1939 by Kurt Gödel (it is possible to add the axiom of choice without causing contradiction) and by Paul Cohen in 1963 (it is also possible to add the negation of the axiom of choice without introducing contradiction). These results mean, as with the parallels postulate, that several different universes can be considered.

In the case of geometry, the independence of the parallels postulate proved that non-Euclidian geometries deserved to be studied and that they could even be used in physics: Albert Einstein took advantage of these when, between 1907 and 1915, he worked out his general theory of relativity.

Regarding the axiom of choice, a similar logical conclusion was warranted; the universes where the axiom of choice is not satisfied must be explored and could be useful in physics.

Toward a set-ist revolution?

However, the attitude of mathematicians toward the axiom of choice is different from their current attitude in the field of geometry. In set theory, the dominant point of view consists of maintaining that the axiom of choice is true, neglecting almost completely competing axioms, yet some of these are very interesting. Everything is as though the equivalent of the revolution of non-Euclidian geometries has not yet occurred in set theory, which today is the general foundation on which all mathematics is constructed.

The implicit argument justifying this lack of interest is that the axiom of choice is only rarely important when dealing with simple questions and that when one deals with more subtle questions – like those of analysis – it is preferable to adopt it, because it is truly necessary to establish the fundamental theorems which we cannot do without.

A series of results concerning the theory of graphs, published in 2003 and 2004 by Alexander Soifer of Princeton University and Saharon Shelah, of the University of Jerusalem, should temper our attitude and invite us to greater curiosity for the alternatives offered by the axiom of choice. The observation demonstrated by A. Soifer and S. Shelah should force mathematicians to reflect on the problems of foundations: what axioms must be retained to form the basis of mathematics for physicists and for mathematicians?

For simple problems of coloring graphs, the two mathematicians demonstrated that the number of colors required depends on whether we take the axiom of choice or one of its weaker versions: in one case, two colors allow for the desired coloring scheme, in the other case, a finite number of colors, no matter how large, is insufficient.

It turns out that knowing if the world of sets satisfies the axiom of choice or a competing axiom is a determining factor in the solution of problems that no one had imagined depended on them. The questions raised by the new results are tied to the fundamental nature of the world of sets. Is it reasonable to believe that the mathematical world of sets is real? If it exists, does the true world of sets – the one in which we think we live – allow the coloring of S. Shelah and A. Soifer in two colors or does it require an infinity of colors?

To grasp the strangeness of the situation created by A. Soifer and S. Shelah's results, let us examine the coloring problems that interest mathematicians.

Coloring a graph

Coloring exercises are among the mathematical problems that even young children face. How can you color a geometric drawing to avoid that two neighboring areas blend together? How do you choose different colors for two contiguous regions? The 4-color theorem provides the answer: no matter how complicated the regions to be colored are – similar to countries drawn on a map – coloring with 4 colors is always possible. Note that this theorem, the proofs of which all use computers, ceases to be true when the map is drawn on a "torus" - the surface of a car tire's inner tube – where some maps require 5, 6, or even 7 colors.

For every map-coloring problem (on a plane, a toric surface, or other) there is a graph obtained by creating a node per country and by linking with an arc any two nodes corresponding the neighboring countries. We study only the coloring of graphs for it is more general: given a graph, we must find a color for each node so that two nodes linked by an arc are of different colors.

Some graphs require two colors, others three, etc. Given a graph, finding the coloring scheme with the most economical number of colors is a difficult problem with a very long calculation time (like the problem of determining divisors for a composite number).

If you can color a finite graph with K colors, while you cannot with K–1 colors (still requiring that nodes that are linked to carry different colors), the graph has chromatic number K. **Back Note 2** shows graphs with chromatic numbers of 2, 3, 4, etc. Finding the chromatic number for a finite graph is only a matter of (long) patience, as it simply requires trying all possible coloring schemes.

If a problem is finite (i.e., if the number n of graph's nodes is finite) and if it can be solved using the axiom of choice, then it can also be solved without using it: it is therefore impossible to find a finite graph whose chromatic number depends on the axiom of choice. The axiom of choice is unavoidable only when infinity appears somewhere in the problem. We did not imagine that the solution to chromatic number problems, which are closer to arithmetic than analysis, could be different depending on whether the axiom of choice was accepted or not. That is the recent discovery by S. Shelah and A. Soifer.

The simplest graph, A. Soifer and S. Shelah's graph G_1, that poses a problem, is the graph defined as follows: the nodes in G_1 are the real numbers (points on a straight line) and the arcs are the pairs (x, y) such that that $(x - y - \sqrt{2})$ is a rational number, in other words, the quotient of two whole numbers.

We cannot draw the graph G_1 by A. Soifer and S. Shelah completely (since it is infinite), but visualizing it does not require too much effort. For example there is no arc between $x = 3$ and $y = 1$, nor between $x = \sqrt{2}$ and $y = \sqrt{3}$, because it's clear in each case that $x - y - \sqrt{2}$ is not rational. On the other hand, there are arcs linking $x = \sqrt{2}$ and $y = 4/5$, or linking $x = \pi + \sqrt{2}$ and $\pi + 3/37$, etc. While the graph G_1 is infinite, it is not complicated, because for a given x and y, we can easily answer the question: "Are they linked in G_1?" Although G_1 is infinite, it does not seem unreasonable to consider that its coloring is a simple problem, quite far from the subtleties of the analysis of differential equations or of theorems of topology of infinite dimensions.

Two colors or an infinite number?

A. Soifer and S. Shelah prove through a demonstration – using the axiom of choice in a general form, denoted *AC* – (see **Back Note 3**) that G_1 can be colored using two colors: if *AC* is true, then the chromatic number for G_1 is 2. On the other hand, they show that G_1 has an infinite chromatic number when you replace the axiom of choice by the combination of the two classic axioms *DC* + *LM* considered as convenient as *AC* for developing the analysis.

The situation is extremely troubling since the two systems under consideration *ZF* + *AC* (Zermelo-Fraenkel set theory with the axiom of choice) and *ZF* + *DC* + *LM* (system proposed by

Solovay in 1964 that allows the appropriate development of the greater part of mathematics needed in physics) seem equally worthy for everything related to the concrete world.

The assertion that two numbers x and y have a difference of the form of $\sqrt{2} + p/q$ depends in no way on an arbitrary decision on the part of the mathematician, just as the assertion that 22 091 is a prime number does not result from an act that one can do or refuse to do. Consequently, the minimum number of colors necessary to color the nodes of graph G_1, without two linked nodes having the same color, seems to be determined in advance without any freedom being given to the mathematician.

Other graphs G_2 and G_n constructed by S. Shelah and A. Soifer according to similar principles and this time linking the points of the plane or space in n dimensions confirm this puzzling link between axiom of choice and chromatic number. Before these results, it was believed that problems of colorability were simple, clear, and concrete problems, whose solution could not depend on which particular form of the axiom of choice was selected. This is not true!

Note that this is not about taking the axiom of choice or refusing it completely, but rather using it either in its strong form *AC*, or in a weaker form, *DC*, accompanied by a convenient axiom, *LM*, all together *DC* + *LM*, forming a system that appears as reasonable as the one obtained with *AC*. Eliminating *AC* abruptly without replacing it with another axiom is not seriously conceivable, since in order to develop the analysis and construct the mathematics necessary for physics (theory of differential systems, partial differential equations, Banach spaces, etc.), you need at least weak forms of the axiom of choice. When choosing the option *DC* + *LM* in place of *AC*, we find ourselves in a world that could be qualified physically as reasonable as the one obtained with *AC*.

The results by S. Shelah and A. Soifer hit the sensitive heart of applied mathematics: while considering only two reasonable theories (for those who do not want to lose the foundation on which contemporary physical sciences are based), they show that the answer to a basic question depends on the theory chosen.

Is there a truth and where is it?

If we believe that the assertion "the graph G_1 by A. Soifer and S. Shelah requires 2 colors" is either true or false, then it means that one of the two theories $ZF + AC$ or $ZF + DC + LM$ is true and the other one is false. It would be good to know which one. Unfortunately, it is unclear on what criteria to base such a decision, and the large majority of mathematicians today are convinced that it is useless to try to find out which of the two theories is "true."

A similar situation was already encountered concerning the Continuum Hypothesis *CH*, the assertion that there is no infinity of a size between the set of whole numbers *N* and the set of real numbers *R*. Although very specific, the question posed by *CH* – which like *AC* is an axiom that is independent of the other axioms of set theory – does not appear to have a solution. In spite of some recent progress due to Hugh Woodin pointing to the falsity of *CH*, today most mathematicians believe that neither option offered (to adopt or refute *CH*) is better or truer than the other is.

The great logician Kurt Gödel (1906-1978) defended the idea that we had not yet identified all the axioms of set theory and that when we have done so, we will no longer be able to choose between AC and DC + LM (or between *CH* and its negation), because one of the options will be excluded by the additional axioms. However, no axiom has imposed itself so far that discredits either of the two options $ZF + AC$ or $ZF + DC + LM$ considered by A. Soifer and S. Shelah. Today, the notion of missing axioms does not receive much support anymore.

Rather than supporting the position that the graph G_1 can or cannot be colored with two colors and that we will finally know when we have completed the set theory, another option can be offered. This option, which most are likely to adopt, consists of concluding from the results of A. Soifer and S. Shelah that, contrary to what we used to think, the chromatic number of a graph does not have a clear meaning in some cases, and therefore there is no absolute truth about the chromatic number for G_1, for example. The colorability of the graph G_1 is not a real problem with a finite solution that we will finally discover and with which everyone will agree, but rather the illusion of a problem that, in the end, has no meaning and for which we can choose the answer freely.

This conclusion is fairly difficult to accept, and the strangeness of the world of set mathematics joins a large number of oddities (for example the one tied to the Continuum Hypothesis, or the Banach-Tarski Paradox according to which a sphere can be broken down to form several spheres of equal volume, etc.). The strangest thing is perhaps that despite all the trouble encountered by set theory, the majority of mathematicians rely nonchalantly on it to establish both their theories and those destined for use in physics.

Explanation of the difficulty of the Nelson problem

The strangeness in set mathematics shown by A. Soifer and S. Shelah might be more important to mathematics than the Continuum Hypothesis, as it could explain the nature of obstacles encountered for over 50 years concerning the chromatic number of the plane. In 1950, an 18-year old young man, who has since become professor at Princeton University and a member of the Academy of Sciences of the United States, proposed a problem stated with great simplicity: what is the chromatic number of the infinite graph whose nodes are the points of a plane and whose arcs are all the pairs of points *M, N* that are one unit apart?

In common language, this becomes: how many different colors does it take to color each point on the graph in such a way that two points one unit apart are never of the same color? Elementary reasoning (see **Back Note 4**) shows that the chromatic number of the plane, *CP*, is at most equal to 7 (with 7 colors, we know how to do the required coloring task) and that it is at least equal to 4 (3 colors are insufficient). In other words, the chromatic number of the plane is 4, 5, 6, or 7.

Yet, for over half a century people have tried to better understand the value of this chromatic number but have made no progress. Paul Erdös relished this problem of great simplicity; he worked on it and widely publicized it… in vain.

Although the results of A. Soifer and S. Shelah do not apply directly to Nelson's Problem, it is clear that the graphs considered by A. Soifer and S. Shelah are of the same nature. The idea that the impossibility of gaining any new insights into Nelson's graph is due to a problem of logic is therefore being seriously considered today. If the chromatic number for Nelson's graph really depends on the axiom of choice, as is the case for A. Soifer and S. Shelah's graphs, a mathematician will be hard-pressed to maintain that foundational questions, and in particular the problem of additional axioms in set theory, are unimportant.

In set theory, as in geometry, all axiomatic systems are not equal. Thinking carefully about their meaning and the consequences of each one of them, and asking ourselves (as it is done in geometry) what the particular usefulness of this or that axiom is in expressing and addressing issues of mathematical physics, may be relevant once again and could lead – why not – to a revolution of set theories, similar to the revolution in non-Euclidian geometries.

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BACK NOTES

Back Note 1. Two forms of the axiom of choice

The axiom of choice exists in several forms that are added to Zermelo-Fraenkel's (*ZF*) axioms of set theory.

1) Axiom of choice in its general form (*AC*):

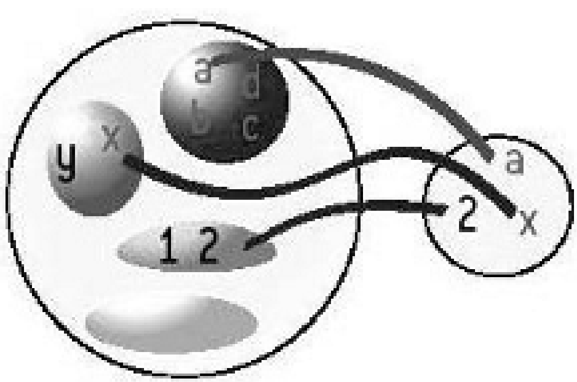


Figure 1

Given a set of disjoint sets that are non-empty, for example E = {{1,2}, {a, b, c}, {x, y}}, then there is a set C composed of one element from each set of E; C = {2, a, y} is suitable for our example (Figure 1). The difficulty appears only with infinite sets.

The axiom *AC* was the subject of many discussions at the beginning of the XXth century for, in spite of its trivial appearance, it results in the existence of objects impossible to construct explicitly: for example, it allows us to prove that there exists an ordering of real numbers that we know we can never construct. It also allows for the decomposition of a 3-dimensional sphere into pieces that recombine to form 2 spheres of identical size to the original sphere (Banach-Tarski Paradox), etc.

2) Axiom of dependent choices (*DC*):

If T is a binary relation (for example defined by a set of arcs) so that for any element *a* of set *A*, there is an element *b*, of *A*, so that *a* and *b* are in relation, denoted *aTb*, then there exists an infinite series *a*(0), *a*(1), …, *a*(*n*) of points in *A*, so that each point is linked by an arc to the next one: *a*(*i*)*Ta*(*i*+1) (Figure 2).

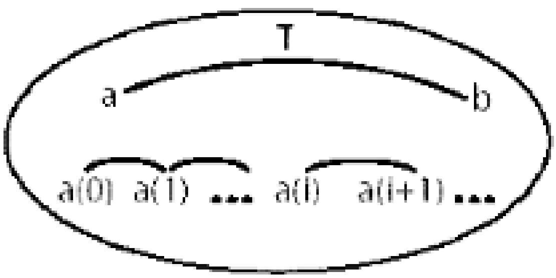


Figure 2

Mathematicians noticed that this axiom *DC*, although less powerful than *AC* (*AC* implies *DC*, but not vice-versa), was enough for most of the demonstrations of analysis needed in mathematical physics. Robert Solovay studied in 1964 the association of *DC* with the axiom *LM* that states that any part of *R* or of R^n is "measurable in the Lebesgue sense" (in the case of R^2 that means that any part of the plane possesses an area).

We know, with *AC*, how to prove the existence of parts of *R* (or of R^n) that are not measurable in the Lebesgue sense. That means that there is incompatibility between *AC* and *LM*. Overall, there are thus two competing theories *ZF* + *AC* and *ZF* + *DC* + *LM*. Each one generally allows us to obtain the same analysis results useful in physics, however they are incompatible: in the first theory, certain parts of the plane have no area; in the second one, any part of the plane possesses an area.

Robert Solovay established that *ZF* + *DC* + *LM* was as reasonable a theory as *ZF* + *AC*. We find ourselves with two competing and incompatible theories, both allowing us to develop mathematics necessary for physics. Which one to choose?

The dilemma between *ZF* + *AC* and *ZF* + *DC* + *LM* looks like the one encountered in the XIXth century between Euclidian geometry, hyperbolic geometry, and elliptical geometry: neither one is more reasonable than the others, they are incompatible, and our world therefore conforms to one of them at most. How do we recognize which one? What is problematic in the case of *ZF* + *AC* and *ZF* + *DC* + *LM* is that A. Soifer and S. Shelah have just demonstrated that for questions of simple coloring of graphs, the two competing theories yield totally contradictory results. The most common attitude until now consisted of maintaining that we did not need to choose between *ZF* + *AC* and *ZF* + *DC* + *LM*: this attitude is no longer tenable.

Back Note 2. The required number of colors

Coloring a drawing or a map is equivalent to coloring a graph: with each country, we can associate a node of the graph and whenever two countries are contiguous, we draw an arc linking the corresponding nodes (Figure 3).

We try to color the nodes of the graph so that no two nodes linked to each other have the same color The smallest number of colors that allows such a coloring is the graph's chromatic number.

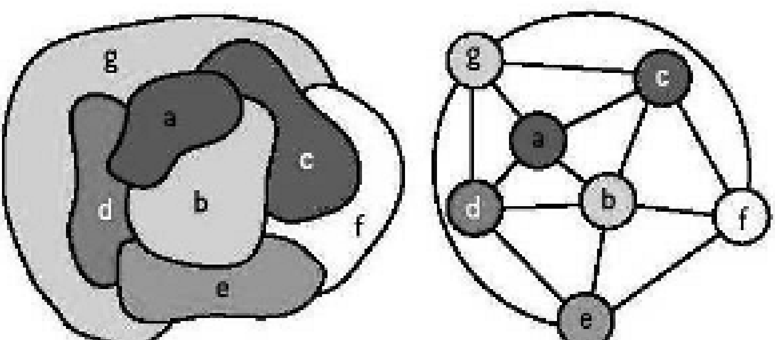
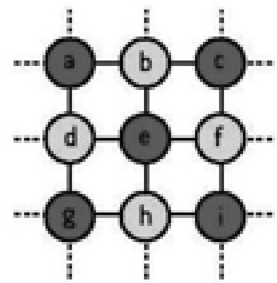
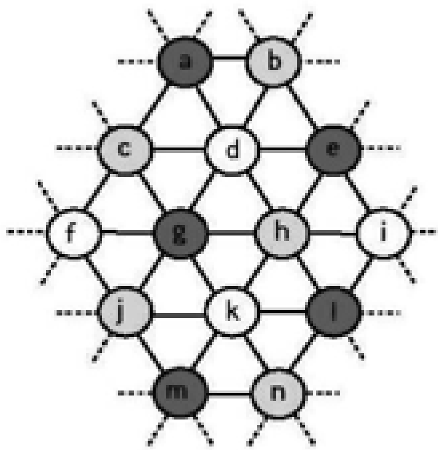


Figure 3

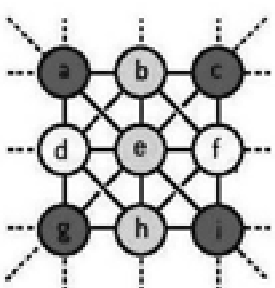
The problem of calculating a finite graph's chromatic number is a complex algorithmic problem: there is no known algorithm that calculates quickly for every graph, and it is believed that none exists.



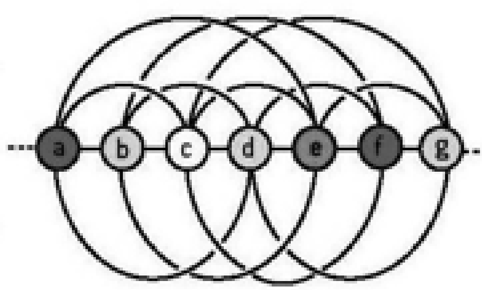
2 colors are needed



3 colors are needed



4 colors are needed



5 colors are needed

Back Note 3. Different chromatic numbers

The surprising result demonstrated by A. Soifer and S. Shelah is as follows:

1. If we accept the axiom of choice in its general form *AC*, then it is possible to color G_1 using two colors without having two points linked by an arc of the same color (the chromatic number of G_1 is 2).

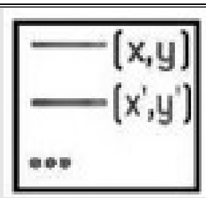
2. Using the statement that all sets of *R* are measurable in the Lebesgue sense (axiom *LM*, compatible with the *DC* form of the axiom of choice) proves that the chromatic number of G_1 cannot be finite and therefore is not 2.

Result 1

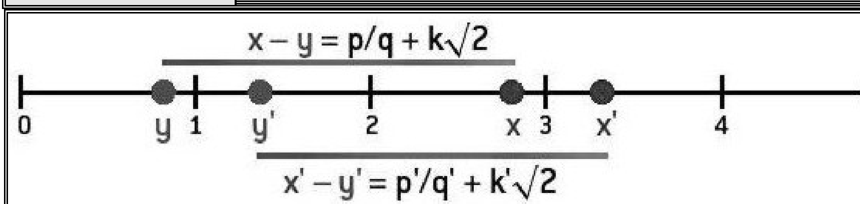
With *AC*, it is possible to color G_1 with two colors.

<u>Idea of the proof</u>

Let T be the relation between points of the line *R* (real numbers) defined by $x\mathrm{T}y$ if $x - y$ is the sum of a rational number p/q and a multiple of $\sqrt{2}$ ($x = y + p/q + k\sqrt{2}$).



Two points x and y are, by definition, in the same equivalence class if $x - y$ is in the form $p/q + k\sqrt{2}$; p, q, k being whole numbers. If x is given, its equivalence class is the set of values y in the form $x + p/q + k\sqrt{2}$; p, q, k taking on all possible whole values. The class for 0 is the set of numbers $p/q + k\sqrt{2}$, the class for π is the set of numbers in the form $\pi + p/q + k\sqrt{2}$, etc.



This relation T combines points of the line *R* into clusters: T is what is called an equivalence relation and the clusters determined by T are the equivalence classes. Therefore, all numbers in the form $p/q + k\sqrt{2}$ (*p, q, k* being any three whole numbers) make up an equivalence class, since each one is linked to the other by the relation T. Here is an example of an equivalence relation: "Being a citizen of one same country. If you start from a French person, you will obtain all French people. If you start from a British person, you will obtain all British people." We can reason on the equivalence class by selecting a representative (as a representative of the equivalence class of the French citizens, we can select the President of the Republic).

The axiom of choice *AC* applied to the set of equivalence classes of T allows us to associate a representative to each equivalence class. *AC* selects a specific element from each class and therefore allows us to associate to each number x, the representative of the class k to which x belongs, namely $f(x)$. By definition, since $f(x)$ is an element of the class, $x = f(x) + p/q + k\sqrt{2}$; where p, q, k are whole numbers.

The coloring in two colors of G_1 is then defined. To determine the color of x, knowing the representative $f(x)$, we can write $x = f(x) + p/q + k\sqrt{2}$. Therefore, if k is even, we color x in blue, if k is odd, we color x in red. It is a theoretical coloring, because in general we cannot construct the value of k, but we know that this whole value exists.

Let us show that if x and y are linked in the graph G_1 of A. Soifer and S. Shelah, then one is red and the other is blue. According to the definition of G_1, if x and y linked in G_1, we have: $x - y = a/b + \sqrt{2}$. Since x and y have the same representative $f(x) = f(y) = f^*$ because of the previous formula, $x = c/d + f^* + e\sqrt{2}$, $y = g/h + f^* + k\sqrt{2}$, with a, b, d, e, g, h, k whole numbers. We gather from this: $e\sqrt{2} - k\sqrt{2} = \sqrt{2} + i/j$ (i, j whole numbers) and therefore necessarily $e - k = 1$. The whole numbers e and k have different parities, which means that x and y are colored differently, which is what we wanted to establish. The coloring obtained through the axiom of choice *AC* is therefore proper and shows that the chromatic number of G_1 is 2.

Result 2

Using *DC* + *LM*, we establish that no coloring with n colors is possible for G_1 (the chromatic number for G_1 is infinite).

<u>Idea of the proof</u>

Suppose we could color G_1 with n colors. Let us call C_1 the set of real numbers of color 1, C_2 the set of real numbers of color 2, etc.

These sets according to the *LM* axiom are measurable in the Lebesgue sense, i.e., they can be attributed a null or a positive length. A reasoning that is relatively simple but that we will not develop here, shows that if a measurable set is of a non-null measure (i.e. if it has a "length", like the segment made up of points between 0 and 1), then it contains at least two elements x and y linked in G_1 .

Consequently, if a measurable set does not contain two numbers linked in graph G_1 , then it is without length. The sets C_1 , C_2 , …, C_n , which by definition never contain two linked points, are therefore without length. Their union is therefore also without length, and so it is impossible for their union to yield R in its entirety. In other words, it is not true that C_1 , C_2 , …, C_n is a coloring of G_1 : we cannot color G_1 with n colors, no matter how large n is.

Back Note 4. Nelson's Problem

We have not made much progress in solving the enigma posed by Edward Nelson in 1950 that could be linked to the axiom of choice: how many different colors does it take to color each point of the plane so that two points one unit apart (for example one centimeter) are never the same color? The pointillist painting by Pissarro illustrates this question.

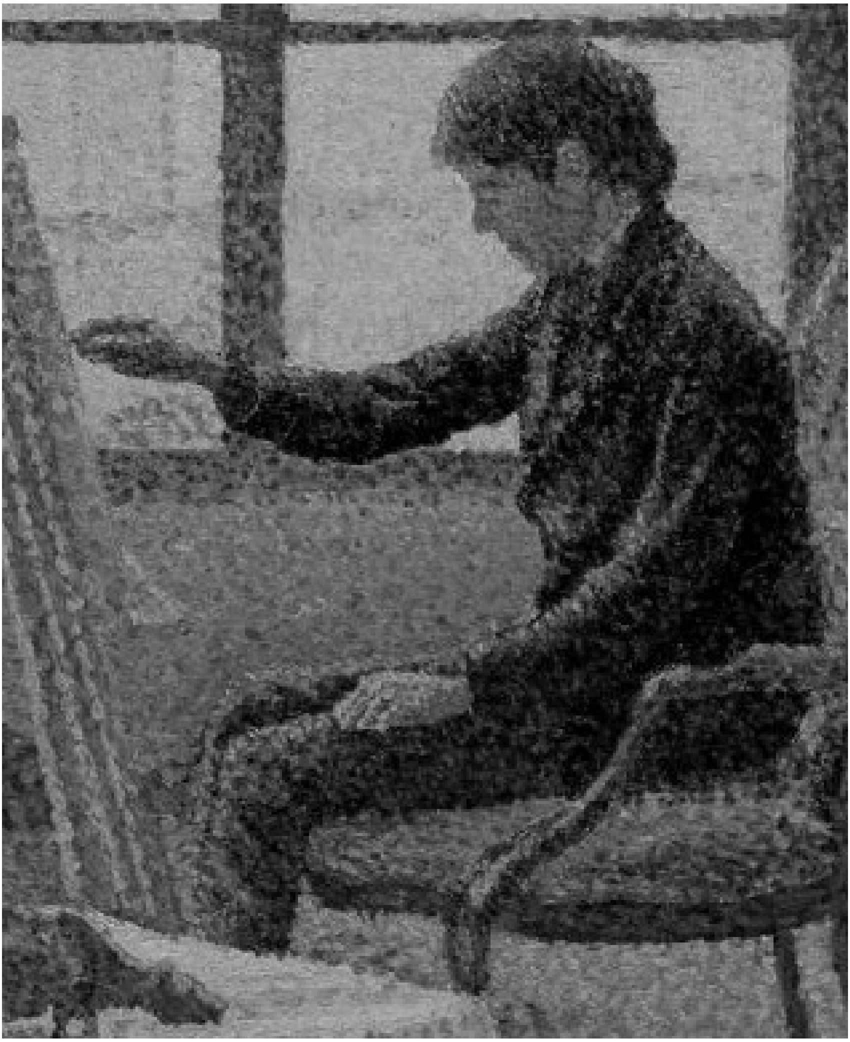
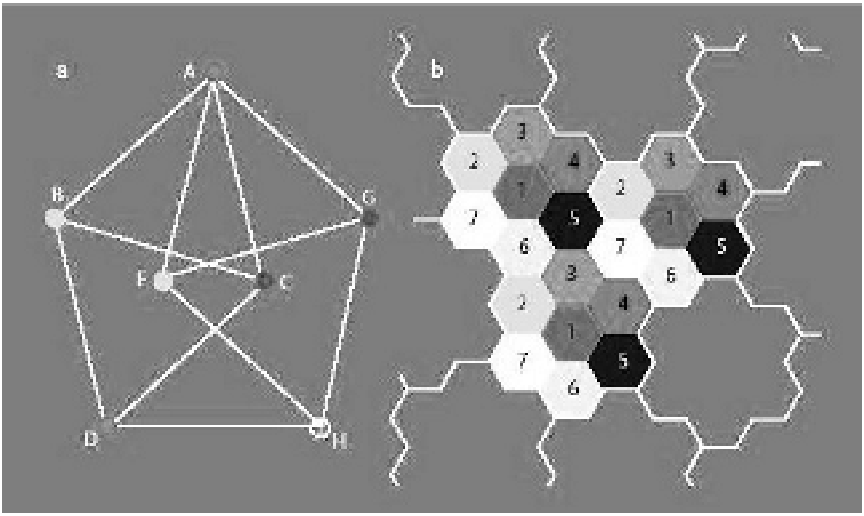


Diagram *a* shows that 3 colors are not enough: the pair B and C and the pair E and G would have 2 different colors, point D must be the same color as A as well as H, and therefore points D and H, although one unit apart, will have the same color. Diagram *b*, where the pairs of points one unit apart are of different colors (the length of the side of the hexagon is 2/5), shows that 7 colors are enough.



No one has yet specified the value of this smallest number that can be 4, 5, 6, or 7.